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Twisted Polynomial Rings Satisfying a Polynomial Identity

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INTRODUCTION

A prime ring is called *right bounded* if every essential right ideal contains a non-zero two-sided ideal. A ring is called *fully bounded* if modulo every prime ideal the ring is right and left bounded.

It is known that even if R is a fully bounded noetherian ring (*FBN ring*) the polynomial ring $R[x]$ may fail to be an FBN ring (for an example, see [6, p. 175]). Cauchon [1] has given necessary and sufficient conditions for R so that $R[x]$ is FBN. In this paper we study this and related questions concerning a generalized version of the polynomial ring, namely, the twisted polynomial ring. All rings are assumed to be associative with an identity. The center of a ring R is denoted by $Z(R)$.

Let σ be a unital ring endomorphism of R. The (*right*) *twisted polynomial* ring $R[x, \sigma]$ consists of polynomials of the form $\sum_{i=0}^{n} x^i a_i$ where $a_i \in R$. Multiplication is defined by requiring associativity, distributivity and the relation $ax = x\sigma(a)$. If we assume R is right (left) noetherian, then it can be shown that σ an automorphism implies $R[x, \sigma]$ is right (left) noetherian. However, for an arbitrary endomorphism σ, $R[x, \sigma]$ may fail to be noetherian on either side (for instance, see [4]). We first examine necessary and sufficient conditions for a twisted polynomial ring to be FBN. Although we are unable to completely classify when $R[x, \sigma]$ is FBN, we are able to obtain a useful classification for an important subfamily of FBN rings, namely, noetherian PI rings. In particular, we determine precisely when $R[x, \sigma]$ is a noetherian PI ring in case σ is an automorphism (see Corollary 10). In Section 2, we discuss conditions under which twisted polynomial rings have "enough" localizable sets of prime ideals in the sense of Müeller [12].

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1. Fully Bounded Noetherian Twisted Polynomial Rings

In this section, both necessary and sufficient conditions as to whether a twisted polynomial ring is FBN or satisfies a polynomial identity (PI) are presented. We first state the following proposition.

Proposition 1. *Let S be a unital subring of a ring T such that ${}_ST$ is a finitely generated free left S-module and let $H = \operatorname{End}({}_ST)$. If there exists a basis for T over S containing the identity element 1_T, then T is a unital subring of H such that H_T is a finitely generated free right T-module with a basis containing 1_T.*

Proof. Since $H = \operatorname{End}({}_ST) = \operatorname{Hom}_S({}_ST, {}_ST_T)$, we have a natural right T-module structure on H. If $\varphi \in H$, $t \in T$, then $\varphi \cdot t$ is defined by the following relation:

$$(\varphi \cdot t)(x) = \varphi(x)\, t \qquad \forall\, x \in T.$$

Thus, we can define a unital ring monomorphism as follows.

$$\theta\colon T \to H$$

$$\theta\colon t \mapsto 1_H \cdot t.$$

We remark that through this ring monomorphism, we can identify 1_T and 1_H.

Let $\{1_T = a_0, a_1, \ldots, a_n\}$ be a basis for ${}_sT$. Let $\psi_i \in H$, $i = 0, 1, \ldots, n$, be defined by

$$\psi_0 = 1_T$$

and

$$\psi_i(a_j) = \delta_{ij}, \qquad i = 1, \ldots, n$$

where $\delta_{ij} = 1_T$ if $i = j$ and 0 otherwise. We claim that $\{1_T = \psi_0, \psi_1, \ldots, \psi_n\}$ is a basis for H_T.

If $\varphi \in H$, then it can be checked that $\varphi = \sum_{i=0}^n \psi_i t_i$ where $t_0 = \varphi(a_0) = \varphi(1_T)$ and $t_i = \varphi(a_i) - a_i t_0$, $i = 1, \ldots, n$. If $\sum_{i=0}^n \psi_i t_i = 0$ for some $t_i \in T$, $i = 0, 1, \ldots, n$, then $t_j = 0$, $j = 0, 1, \ldots, n$, since $0 = \sum_{i=0}^n (\psi_i \cdot t_i)(a_j) = t_j$. ∎

If we restrict this proposition to the case where $T = R[x, \sigma]$ is a twisted polynomial ring, we obtain the following useful corollary.

Corollary 2. *If σ is an automorphism of the ring R, then the following are equivalent*:

(i) *$R[x, \sigma]$ is an* FBN *ring.*

(ii) *$R[x, \sigma^n]$ is an* FBN *ring for some integer n.*

Proof. (ii) $\Rightarrow$ (i) Let $T = R[x, \sigma]$ and $S = R[x, \sigma^n]$ for some integer n. There is an embedding from S to T given by $x \to x^n$. The rings T and S satisfy the hypotheses of Proposition 1. In particular, the elements $\{1_R = 1_T, ..., x^{n-1}\}$ form a basis of T as a right and left S-module.

We need to know that, in general, the matrix ring over an FBN ring is again FBN. To that end, let A be an arbitrary FBN ring and let $B = M_n(A)$, the ring of $n \times n$ matrices over A. Clearly, B is noetherian. We might as well assume A is prime, so B is also prime. Thus B has a simple classical quotient ring $M_n(Q)$ where Q is the quotient ring of A. Let I be an essential right ideal of B. Then I must contain a regular element C which is invertible in $M_n(Q)$ by a matrix D. The elements of D are of the form $a_{ij}q_{ij}^{-1}$, but by common denominators the q_{ij} may be replaced by a single q. So $CD = 1$, $C(Dq) = 1q$ and $Dq \in B$. Thus $I \cap A$ contains the regular element q. Since A is FBN, $I \cap A$ contains a non-zero two-sided ideal J. Observe that $JB = M_n(J)$ is a two-sided ideal of B contained in I and so B is fully bounded.

Returning to our particular case we see that $H = \mathrm{End}(_S T) \simeq \mathrm{End}(T_S)$ is an FBN ring. By [1, Proposition 2.1] T is an FBN ring. ∎

(i) $\Rightarrow$ (ii) This follows immediately from [1, Proposition 2.1]. ∎

Later in this section, we examine a subfamily of FBN rings, namely, noetherian PI rings. For these rings the following corollary to Proposition 1 is useful. Let S and T be defined as in Proposition 1.

COROLLARY 3. *S satisfies a polynomial identity if and only if T satisfies a polynomial identity.*

Proof. The result follows immediately from the fact that a subring of a PI ring and an $n \times n$ matrix ring over a PI ring are themselves PI rings. ∎

We use these results to attain necessary criteria for $R[x, \sigma]$ to be an FBN ring. Let R be any noetherian ring and let σ be an automorphism of R. If N is the nilradical of R, one can easily check that $\sigma(N) = N$. Therefore there is an induced automorphism

$$\bar{\sigma}: R/N \to R/N.$$

Moreover, if $Z(R/N)$ is the center of R/N, $\bar{\sigma}$ can be restricted to this ring. We denote this restriction by $\bar{\sigma}_{Z(R/N)}$.

PROPOSITION 4. *Let R be a noetherian ring with σ an automorphism on R. If $R[x, \sigma]$ is an* FBN *ring, then there exists an integer m such that $\bar{\sigma}^m_{Z(R/N)} = 1_{Z(R/N)}$.*

Proof. First, we assume R is a prime noetherian ring. It follows easily that $R[x, \sigma]$ is also a prime ring. Note that the monic polynomial $x + 1$ is a regular element of $R[x, \sigma]$. Regular elements in a prime noetherian ring generate essential right ideals. Hence, since $R[x, \sigma]$ is FBN, the right ideal generated by $x+1$ must contain a non-zero two-sided ideal I. Let $f(x) = x^n a_n + x^{n-1} a_{n-1} + \cdots + a_0$ be a non-zero element of minimal degree in I. Since $x+1$ divides $f(x)$, we know that at least one of the non-leading coefficients of $f(x)$, say, a_i, $i \neq n$, is non-zero. If b is any element of $Z(R)$, then $g(x) = b \cdot f(x) - f(x) \cdot \sigma^n(b)$ is an element of I. Since $\sigma^n(b)$ is a central element $g(x)$ has degree less than $f(x)$. Thus $g(x) = 0$. In particular, since $a_i \neq 0$ and since $\sigma^i(b) - \sigma^n(b)$ is in the center of the prime ring R, it follows that $\sigma^i(b) = \sigma^n(b)$. Therefore, if $m = n - i$, we see that $\sigma^m_{Z(R)} = 1_{Z(R)}$.

Now let R be a semiprime noetherian ring. By Goldie's theorem [15, II, 2.4] there exists a finite number of minimal prime ideals $P_1, ..., P_k$ of R. Since σ is an automorphism, $\sigma(P_i) = P_j$ for some j. Since the σ acts as a permutation on the set $\{1, ..., k\}$, there exists an integer n such that $\sigma^n(P_i) = P_i$ for all i. If $\rho = \sigma^n$, then by Corollary 2 we know that $R[x, \rho]$ is FBN. Therefore $R[x, \rho]/P_i R[x, \rho] \cong R/P_i[x, \bar{\rho}]$ is a prime FBN ring. Hence, there exist an m_i such that $\bar{\sigma}^{m_i}_{Z(R/P_i)} = 1_{Z(R/P_i)}$. Let m equal the least common multiple of the set $\{nm_i \mid i = 1, ..., k\}$. It is clear that $\sigma^m(P_i) = P_i$ and $\bar{\sigma}^m_{Z(R/P_i)} = 1_{(R/P_i)}$ for each $i = 1, ..., k$. If $b \in Z(R)$, then the image of b is central in each R/P_i. Therefore $b - \sigma^m(b) \in P_i$ for each i. Since $P_i \cap \cdots \cap P_k = 0$, $\sigma^m(b) = b$ and we are done.

If R is any noetherian ring with nilradical N, then the proposition follows since $R[x, \sigma]/N[x, \sigma] \cong R/N[x, \bar{\sigma}]$. ∎

We remark that although we required a noetherian hypothesis, similar results are attainable if the ring has a finite set of minimal prime ideals with nilpotent nilradical. In the case of a ring satisfying a polynomial identity, we can prove a stronger result. We recall that given a semiprime ideal N of R, then $\mathscr{C}(N)$ will denote the elements of R which are regular modulo N.

LEMMA 5. *If R is a semiprime, right Goldie ring and if σ is a monic endomorphism of R such that $\sigma(\mathscr{C}(0) \subseteq \mathscr{C}(0)$, then for each minimal prime ideal P of R, there exists an integer n such that $\sigma^n(P) \subseteq P$.*

Proof. We first consider the case when R is a semisimple artinian ring. Let $P_1, P_2, ..., P_k$ be the prime ideals of R and let $\{e_1, e_2, ..., e_k\}$ be a complete set of orthogonal primitive idempotents of R. Therefore $\{\sigma(e_1), ..., \sigma(e_n)\}$ is also a complete set of orthogonal primitive idempotents. By [9, p. 54, Theorem 3], there exists a unit $u \in R$ such that $e_i = u^{-1}\sigma(e_i)\, u$. Let π^u be the automorphism of R that is defined via conjugation by u. Thus the endomorphism $\pi^u \circ \sigma$ fixes each e_i. Since any ideal of R is generated by a subset of $\{e_1, ..., e_n\}$ as a right ideal, it follows that $\pi^u \circ \sigma(P_i) \subseteq P_i$. Since $\pi^{u^{-1}}$ is an automorphism of R, we know that $\pi^{u^{-1}}(P_i) = P_j$ for some j. Therefore, $\sigma(P_i) = \pi^{u^{-1}} \circ \pi^u \circ \sigma(P_i) \subseteq \pi^{u^{-1}}(P_i) = P_j$.

Since $\pi^u \circ \sigma$ fixes each e_i, $\pi^u \circ \sigma(P_i) R = P_i$. Thus, if $i \neq k$, $\pi^u \circ \sigma(P_k) \not\subseteq P_i$ and it follows that $\sigma(P_i)$ and $\sigma(P_k)$ are not contained in the same P_j for i unequal to k. By a counting argument, we see that for each j, P_j contains $\sigma(P_i)$ for one and only one i. More generally, for each integer m, P_j contains $\sigma^m(P_i)$ for one and only one i. Clearly for each prime ideal P there exists distinct integers m and t such that $\sigma^m(P) = \sigma^t(P)$ are contained in the same prime ideal. Thus $\sigma^{m-1}(P)$ and $\sigma^{t-1}(P)$ also must be contained in the same prime ideal. By continuing inductively in this manner, we have the special case.

Suppose R is a semiprime Goldie ring with endomorphism σ. It follows that the classical ring of quotients Q is a semisimple ring with prime ideals $\bar{P}_1, ..., \bar{P}_k$ which are the localizations of the minimal prime ideals $P_1, ..., P_k$ of R. Moreover, since σ takes regular elements to regular elements it can be extended to a monic endomorphism of Q. The result now follows. ∎

We now derive a stronger version of Proposition 4 in the case of PI ring.

PROPOSITION 6. *Let R be a semiprime, right Goldie ring with σ a monic endomorphism such that $\sigma(\mathscr{C}(0)) \subseteq \mathscr{C}(0)$. If $R[x, \sigma]$ is a PI ring, then an integer n exists such that $\sigma^n_{Z(R)} = 1_{Z(R)}$.*

Proof. Assume R is a prime ring. It follows that $R[x, \sigma]$ is a prime ring and thus the two-sided ideal $xR[x, \sigma]$ contains a non-zero central element $f(x)$ [13, Theorem 1.7.11]. We note that $f(x)$ must be of degree at least one. Let $f(x) = x^n a_n + x^{n-1} a_{n-1} + \cdots + x a_1$. Since $f(x)$ is central, it follows that each monomial $x^i a_i$ is central. In particular $x^n a_n$ is central. If $b \in Z(R)$, then $b x^n a_n = x^n a_n b$. This implies $\sigma^n(b) a_n = b a_n$ and hence $(\sigma^n(b) - b) a_n = 0$. However, since $x^n a_n$ is a central element of a prime ring it must be a regular element of that ring. It follows that a_n is a regular element of R. Thus $\sigma^n(b) = b$.

Let R be semiprime. By Lemma 5, there exists an integer n such that for every minimal prime ideal P of R, $\sigma^n(P) \subseteq P$. Thus $\hat{P} = \sum x^i P$ is an ideal of $R[x, \sigma^n]$. If $\bar{\sigma}^n$ is the induced endomorphism of R/P, then $R[x, \sigma^n]/\hat{P} \cong R/P[x, \bar{\sigma}^n]$. The ring $R/P[x, \bar{\sigma}^n]$ is a PI ring. We claim $\bar{\sigma}^n$ is a monomorphism. Suppose $\bar{\sigma}^n(P) \supset P$, then $\bar{\sigma}^n(P)$ must contain a regular element. Since σ takes regular elements to regular elements, so does σ^n. Hence P must contain a regular element. However, P is a minimal prime ideal of a semiprime, Goldie ring, so it is the annihilator of an ideal of R, a contradiction. Thus $\bar{\sigma}^n(P) = P$ and $\bar{\sigma}^n$ is a monomorphism. Hence, $R/P[x, \bar{\sigma}^n]$ is a prime ring. We have reduced the problem to that of a prime PI ring and the result follows as in Proposition 4. ∎

So far, we have produced necessary conditions for a twisted polynomial ring to be FBN or to satisfy a polynomial identity. In Propositions 7, 8 and

9 we give sufficient conditions for $R[x, \sigma]$ to be FBN or to satisfy a polynomial identity. While we do not have a complete classification in the FBN case, we do have the following result in that direction.

PROPOSITION 7. *Let R be an* FBN *ring with nilradical N such that $R[x]$ is an* FBN *ring. If σ is an automorphism of R such that $\bar{\sigma}^n = 1_{R/N}$ for some integer n, then $R[x, \sigma]$ is an* FBN *ring.*

Proof. Since $N[x, \sigma]$ is nilpotent and hence contained in every prime ideal, it suffices to prove that $R[x, \sigma]/N[x, \sigma] \cong R/N[x, \bar{\sigma}]$ is an FBN ring. But $R/N[x, \bar{\sigma}^n] \cong R/N[x]$ is an FBN ring. By Corollary 2, $R/N[x, \bar{\sigma}]$ is an FBN ring. ∎

For a complete classification of when $R[x]$ is an FBN ring, we refer the reader to a paper of Cauchon [1]. Recall that an important class of FBN rings are those noetherian rings algebraic over their center. If R is such a ring, then so is $R[x]$. Hence, Proposition 7 is applicable is this case. For these rings, we can attain a result somewhat more general result than Proposition 7.

PROPOSITION 8. *Let R be a noetherian ring algebraic over its center with nilradical N. If σ is an automorphism of R such that the induced map $\bar{\sigma}^n$ is conjugation by some unit $\bar{u}$ in R/N, then $R[x, \sigma]$ is an* FBN *ring.*

Proof. As in Proposition 7, we can assume R is a semiprime ring and that σ is conjugation by a unit u of R. Since σ is an automorphism, $R[x, \sigma]$ is noetherian [11]. We first claim that $R[x, \sigma]$ is algebraic over its center. We note that a monomial of the form $x^m u^{-m} a$ with $a \in Z(R)$ is central in $R[x, \sigma]$. Clearly the elements of R are algebraic over the center of $R[x, \sigma]$. Thus we will be done if we can show x satisfies a polynomial of the form $y^n a_0 + y^{n-1} x u^{-1} a_1 + y^{n-2} x^2 u^{-2} a_2 + \cdots + x^n u^{-n} a_n$ where each $a_i \in Z(R)$. If we substitute x for y in this equation, we obtain the relation

$$x^n(a_0 + u^{-1} a_1 + \cdots + u^{-n} a_n) = 0.$$

Thus, it suffices to show that there exist elements $a_i \in Z(R)$ such that $a_0 + u^{-1} a_1 + \cdots + u^{-n} a_n = 0$. However, if we multiply this equation by u^n, we see that it suffices to show u satisfies a polynomial over $Z(R)$. Since R is algebraic over $Z(R)$ we have the claim. Finally, we want to show that if a ring T is a noetherian ring algebraic over its center Z, then it is an FBN ring. Without loss of generality assume T is prime and algebraic over Z. Let I be an essential right ideal of R, we must show that I contains a non-zero two-sided ideal. By Goldie's Theorem, I contains a regular element c. Let $p(x) = x^n a_n + x^{n-1} a_{n-1} + \cdots + x a_1 + a_0$ be the polynomial over Z of least degree satisfied by c. If $a_0 \neq 0$ then $a_0 \in I \cap Z$ so I contains a two-sided ideal. If

$a_0 = 0$, then $c(c^{n-1}a_n + c^{n-2}a_{n-1} + \cdots + a_1) = 0$ and since c is regular, c satisfies a polynomial of degree smaller then $p(x)$, a contradiction. Thus $R[x, \sigma]$ is FBN. ∎

We now give sufficient conditions for $R[x, \sigma]$ to satisfy a polynomial identity.

PROPOSITION 9. *Let R be a* PI *ring with nilpotent nilradical N such that N is the intersection of finite number of prime ideals. If σ is a monic endomorphism of R such that $\sigma(N) \subseteq N$, $\sigma(\mathscr{C}(N)) \subseteq \mathscr{C}(N)$, and $\bar{\sigma}^n = 1_{Z(R/N)}$ for some integer n, then $R[x, \sigma]$ is a* PI *ring.*

Proof. Since σ is a monic endomorphism, it follows that $\sigma^{-1}(N) \subseteq N$. Thus if $\sigma(N) \subseteq N$, we have an induced endomorphism $\bar{\sigma}$ of R/N which is monic and sends regular elements to regular elements. Again, as in our previous propositions, we can now assume R is semiprime.

Since R has a finite number of minimal prime ideals, R has a semisimple artinian ring of quotients Q [13, Theorem 1.7.34]. It follows that Q is a PI ring, finite dimensional over its center $Z(Q)$, and $Z(Q)$ is the classical ring of quotients of the center of R [13, Theorem 1.7.20].

Let $\hat{\sigma}$ be the extension of σ to Q. We note that $\hat{\sigma}^n = 1_{Z(Q)}$ since $Z(Q)$ is classical ring of quotients of $Z(R)$. Since Q is finitely generated, free over $Z(Q)$ with a basis containing the identity, the same can be said for $Q[x, \hat{\sigma}^n]$ over $Z(Q)[x]$. By Corollary 3, $Q[X, \hat{\sigma}^n]$ satisfying a polynomial identity implies that $Q[x, \hat{\sigma}]$ is a PI ring. Since $R[x, \sigma] \subseteq Q[x, \hat{\sigma}]$, we are done. ∎

COROLLARY 10. *If R is a noetherian* PI *ring with automorphism σ and nilradical N, then the following are equivalent:*

(i) *$R[x, \sigma]$ is a noetherian* PI *ring.*

(ii) *$\bar{\sigma}^n = 1_{Z(R/N)}$ for some integer n.*

Proof. (i) $\Rightarrow$ (ii) If $R[x, \sigma]$ is a noetherian PI ring, then the result follows from Proposition 4.

(ii) $\Rightarrow$ (i) The results follows from Proposition 9. ∎

We now give two examples. In the first, we have a commutative artinian ring R and an automorphism σ such that $\sigma^n \neq 1_R$ for any n, but whose induced automorphism is the identity on R/N. In the second example we give an automorphism σ such that no power of σ is the identity on all of R but σ restricted to the center is the identity. In both cases $R[x, \sigma]$ satisfies a polynomial identity by the results of this section.

EXAMPLE 1. Let $R = \{\left(\begin{smallmatrix} a & 0 \\ b & a \end{smallmatrix}\right) \mid a, b \in \mathbb{C}$, the complex numbers$\}$. Let u be a non-zero element of $\mathbb{C}$ which is not a root of unity. Define

$$\sigma: R \to R$$
$$: \begin{pmatrix} a & 0 \\ b & a \end{pmatrix} \to \begin{pmatrix} a & 0 \\ ub & a \end{pmatrix}.$$

This is an automorphism, no power of which is the identity. It is clear that $\bar{\sigma}$ is the identity on R/N.

EXAMPLE 2. Let R be the ring of 2×2 matrices over the rationals and let $u = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. Let σ be conjugation by u. Of course $\sigma = 1_{Z(R)}$ and no power of σ is the identity.

To find a noetherian, PI ring R and an automorphism σ such that $R[x, \sigma]$ is not a PI ring one need only take a commutative field R and an automorphism σ of infinite order.

2. LOCALIZATION OF TWISTED POLYNOMIAL RINGS

Recall that a finite set $\{P_1, P_2, ..., P_n\}$ of prime ideals is said to be *localizable* if $\mathscr{C}(N)$ is an Ore set where $N = \bigcap_{i=1}^{n} P_i$. In an FBN ring, Müeller [12] calls a minimal localizable set of prime ideals a *clan*. He showed that a prime ideal is contained in at most one clan. We will use [P] to denote the unique clan containing P (when the clan exists). If every prime of an FBN ring belongs to a localizable set we say the ring has *enough clans*. Moreover, if R is an FBN ring with enough clans such that each clan is a singleton set, we say R is *stable* [3].

From the first section we know that $R[x, \sigma]$ will be FBN under certain circumstances. In this section we want to discuss when primes ideals of $R[x, \sigma]$ are localizable.

PROPOSITION 11. *Let R be a simple artinian ring with automorphism σ. If $R[x, \sigma]$ is* FBN, *then $R[x, \sigma]$ is stable.*

Proof. We note that $R[x, \sigma]$ has global dimension 1, i.e., $R[x, \sigma]$ is a hereditary ring [5, Corollary]. As we said before, it is easy to see that $R[x, \sigma]$ is a prime ring. Therefore, it follows from [4, Proposition 2.2] and [2, Theorem 3.3] that we need only show $R[x, \sigma]$ has no idempotent non-zero two-sided ideals. It is known that $R[x, \sigma]$ is a principal right ideal ring (for instance, see [10, Theorem 3.1]). So, if I is two-sided ideal, then $I = f \cdot R[x, \sigma]$ for some $f \in R[x, \sigma]$. If $I^2 = I$, then $f \cdot R[x, \sigma] \subseteq fR[x, \sigma] \cdot fR[x, \sigma] \subseteq f^2 \cdot R[x, \sigma]$. Thus $f = f^2 g$ for some $g \in R[x, \sigma]$. Therefore, either $f \cdot g = 1$, a contradiction, or f is a zero divisor. However, $f \cdot R[x, \sigma]$ is an ideal of a prime ring, so it must contain a regular element. Hence, f itself must be regular. Thus $I^2 \neq I$ and we are done. ∎

We will need some terminology before going to the next result. The Krull dimension of a ring R, in the sense of Gordon and Robson [7], will be denoted K-dim R. If R is an FBN ring then K-dim R is the same as the classical Krull dimension of R. If M is an R module, $E(M)$ will denote the injective full of M.

Recall that a ring S is called a finite normalizing extension of a unital subring R provided that S is finitely generated as an R-module by elements which normalize R; that is, $S = \sum_{i=1}^{n} Ra_i$ with $Ra_i = a_iR$ for each i.

THEOREM 12. *Let $S \subseteq T$ be a finite normalizing extension of S such that T is free as a left and right S-module. Suppose a normalizing basis for T over S is of the form $\{1, b, b^2, ..., b^{n-1}\}$ where b is regular in T with $b^nQ = Qb^n$ for all primes Q in S. If S is* FBN *with enough clans, then T is* FBN *with enough clans.*

Proof. By Proposition 1 and [1, Proposition 2.11] T is an FBN ring. Let P be a prime ideal of T. By Heinicke and Robson [8, Corollary 2.12], there exist a finite number of prime ideals of S, $\{Q_1, ..., Q_r\}$ minimal over $P \cap S$. Moreover, $S/Q_i \cong S/Q_j$ $1 \leqslant i, j \leqslant r$, and for each i, $1 \leqslant i \leqslant r$, there exists j, $1 \leqslant j \leqslant n-1$, such that $\bar{Q}_i \bar{b}^j = \bar{b}^j \bar{Q}_1$ where $\bar{X}$ denotes the image of the set X in T/P [8, Theorem 2.11].

Since b is regular in T and $bS = Sb$, b induces a ring automorphism σ of S defined by $sb = b\sigma(s)$. Thus

$$\bar{Q}_i \bar{b}^j = \bar{b}^j \bar{Q}_1 = \overline{\sigma^j(Q_1)} \cdot \bar{b}^j.$$

If $b \notin P$, then $\bar{b}$ is a normalizing element of T/P, i.e., $T/P \cdot \bar{b} = \bar{b} \cdot T/P$. Hence $\bar{b}$ is regular in T/P. Therefore, $\bar{Q}_i = \overline{\sigma^j(Q_1)}$ and since $P \cap S \subseteq Q_i$ it follows that $\sigma^j(Q_1) \subseteq Q_i$. However, $K\text{-dim}(S/\sigma^j(Q_1)) = K\text{-dim}(S/Q_1) = K\text{-dim}(S/Q_i)$, and hence $\sigma^j(Q_1) = Q_i$.

If $b \in P$, it is clear that $S/P \cap S = T/P$, so $P \cap S$ is prime. In either case, the prime ideals of S minimal over $P \cap S$ are a subset of the set $\{\sigma^i(Q) \mid Q$ is a minimal prime over $P \cap S$, $0 \leqslant i \leqslant n-1\}$ (note $\sigma^n(Q) = Q$ by hypothesis). Let

$$U_p = \{Q \mid Q \in [\sigma^j(Q_p)] \text{ where } Q_p \text{ is a prime minimal over } P \cap S\}$$

and

$$V_p = \{P' \mid P' \text{ is prime ideal of } T \text{ and there exists } Q \in U_p \text{ minimal over } P'\}.$$

We note that since S has enough clans, U_p is a finite set. By [8, Corollary 2.12], V_p is a finite set. Let $P' \in V_p$. Suppose there exists a T-homomorphism

$$\varphi: E(T/P') \to E(T/P'')$$

where P'' is a prime ideal of T and $K\text{-dim}(T/P') = K\text{-dim}(T/P'')$. In order to show T has enough clans, it suffices to prove that $P'' \in V_p$ [12, Theorem 5].

Let $E_S(T/P')$ be the left S-injective envelope of T/P' and consider the left T-module $\mathrm{Hom}_S({}_ST_T, E_S(T/P'))$. By a modification of [15, Proposition 10.4], $\mathrm{Hom}_S(T, E_S(T/P'))$ is an injective T-module. Consider the quotient homomorphism $\psi: T \to T/P'$. It follows that $P' = \mathrm{ann}_T(\psi)$. Therefore $E(T/P') \subseteq \mathrm{Hom}_S(T, E_S(T/P'))$ as a T-module. As an S-module

$$\begin{aligned}\mathrm{Hom}_S(T, E_S(T/P')) &\cong \bigoplus_{i=0}^{n-1} \mathrm{Hom}_S(Sb^i, E_S(T/P'))\\ &\cong \bigoplus_{i=0}^{n-1} \mathrm{Hom}_S(Sb^i, \bigoplus_{k=1}^{m} E(S/N_k)) \cong \bigoplus_{k=1}^{m} \bigoplus_{i=0}^{n-1} \mathrm{Hom}_S(Sb^i, E(S/N_k))\end{aligned}$$

where $N_k \in \{\sigma^j(Q') \mid Q' \in U_p$ and minimal over $P' \cap S\}$ [8, Theorem 3.6]. One checks that $\mathrm{Hom}_S(Sb^i, E(S/N_k))$ is isomorphic to $E(S/M_k)$ where $M_k \in \{\sigma^j(Q')\}$. Thus $E(T/P') \cong \oplus_{h=1}^{r} E(S/M_k)$ where $M_k \in \{\sigma^j(Q')\}$.

Since $\varphi: E(T/P') \to E(T/P'')$ is a non-zero S-homomorphism, without loss of generality, $\varphi(E(S/M_1)) \neq 0$. Let $E(S/N)$ be an S-direct summand of $E(T/P'')$ such that there exists a non-zero S-homomorphism $E(S/M_1) \to E(S/N)$, N a prime ideal of S minimal over $P'' \cap S$. Since $K\text{-dim}(S/M_1) = K\text{-dim}(T/P') = K\text{-dim}(T/P'') = K\text{-dim}(S/N)$, it follows that $N \in [M_1]$. We note that $M_1 = \sigma^t(Q')$ where $Q' \in U_p$ and $0 \leqslant t \leqslant n-1$. By [12, Proposition 2(4)], one can check U_p is closed under σ. Hence, $M_1 \in U_p$ and it follows that $N \in U_p$. By definition, $P'' \in V_p$. ∎

COROLLARY 13. *Let R be a noetherian ring integral over its center. If σ is an automorphism of R such that σ^n is conjugation by some unit of R then $R[x, \sigma]$ is an* FBN *ring with enough clans.*

Proof. By Proposition 8 $R[x, \sigma]$ is an FBN ring. It follows from the proof of Proposition 8 that $R[x, \sigma^n] \cong R[x^n, \sigma]$ is integral over its center. Thus $R[x^n, \sigma]$ has enough clans [12, Proposition 9]. Clearly $R[x^n, \sigma]$ and $R[x, \sigma]$ satisfy the relationship of S and T, respectively, in Theorem 12. ∎

COROLLARY 14. *Let R be a finitely generated algebra over a field. Let σ be an automorphism of R of finite order. If R is a noetherian* PI *ring with enough clans then so is $R[x, \sigma]$.*

Proof. In [14, Proposition 9] it was shown that the corollary is true if σ is the identity. When combined with Corollary 3 and Theorem 12 we have the desired result. ∎

We note that in the preceding results there exists a non-zero integer j such that $\sigma^j(P) = P$ for each prime P in R. In a sense this property is essential to ensure that $R[x, \sigma]$ has enough clans.

EXAMPLE 3. Let $R = \mathbb{C}[y]/\langle y^2 \rangle$. We define an automorphism

$$\sigma: R \to R$$

$$\sigma: a_0 + a_1 y \to a_0 + a_1 uy$$

where $0 \neq u \in \mathbb{C}$ such that u is not a root of unity. Consider the twisted polynomial ring $R[x, \sigma]$. Let M_c be the left ideal generated by $x - c$ and y, $c \in \mathbb{C}$. Note that y is a normalizing element and $R[x, \sigma]/\langle y \rangle$ is a commutative ring. Hence M_c is a two-sided maximal ideal. We claim that if $0 \neq a = ub$ then M_a and M_b are two maximal ideals which are linked in the sense of Müeller [12]. By [12, p. 236, Remark (2)] it suffices to show $M_a \cap M_b \supsetneq M_a M_b$. Since $y \in M_a \cap M_b$ it will suffice to show that $y \notin M_a M_b$.

Suppose $y \in M_a \cdot M_b$. Therefore

$$\begin{aligned} y &= q_1(x-a)(x-b) + q_2(x-a)y + q_3 y(x-b) \\ &= q_1(x-a)(x-b) + q_2 y(xu^{-1} - a) + q_3 y(x-b) \\ &= q_1(x-a)(x-b) + q_2 yu^{-1}(x-b) + q_3 y(x-b) \\ &= q(x-b) \end{aligned}$$

where q_1, q_2, q_3 and $q \in R[x, \sigma]$. The left side of this equation has degree 0 as a polynomial in x, while the right side has degree at least one, unless $q = 0$. In either case we have a contradiction. Thus M_b and M_{ub} are linked. By induction, M_b and $M_{u^k b}$ are also linked for all $k \in \mathbb{N}$. Since u is not a root of unity, we have infinitely many primes "linked" together. Hence M_b, $b \neq 0$, does not belong to a localizable set of prime ideals [10, Theorem 5].

Given Theorem 12 and [12, Proposition 9] it is natural to ask for what PI rings R does $R[x]$ have enough clans? Our next example, which is a variation of the previous one, shows that even if R is an artinian PI ring with only one non-zero ideal, $R[x]$ may not have enough clans.

EXAMPLE 4. Let F be any field and let $Q = F(z_i)$, $i \in \mathbb{Z}$ be the field of rational functions of F adjoining countably many commuting indeterminates. Let σ be the endomorphism of Q which fixes F and sends z_i to z_{i+1}. Let

$R = Q[y, \sigma]/(y^2)$. It is not difficult to see that R is an artinian PI ring (since modulo the nilradical R is commutative) with unique non-zero ideal (y).

Consider the ring $R[x]$. For each z_i let M_i be the left ideal generated by $x - z_i$ and y. As before M_i is a two-sided maximal ideal. We claim M_i is linked to M_{i+1}. As before it will suffice to show that $y \notin M_i \cdot M_{i+1}$. Assume the contrary. Then

$$\begin{aligned} y &= q_1(x - z_i)(x - z_{i+1}) + q_2(x - z_i)\,y + q_3 y(x - z_{i+1}) \\ &= q_1(x - z_i)(x - z_{i+1}) + q_2 y(x - z_{i+1}) + q_3 y(x - z_{i+1}) \\ &= q(x - z_{i+1}) \end{aligned}$$

where q_1, q_2, q_3 and $q \in R[x]$. As in the previous example we have a contradiction. So again, M_i cannot belong to a clan.

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